Journal of Applied Mechanics and Technical Physics, Vol. 43, No. 2, pp. 269–273, 2002

FLOWS OF NON-NEWTONIAN FLUIDS WITH HYDRAULIC JUMPS

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UDC 532.536

Steady and unsteady waves propagating over the surface of a thin layer of a dilatant fluid moving over an inclined plane, with rheological properties of the fluid described by the Ostwald–de Waele power law, are studied analytically and numerically.

Viscosity of many fluids of natural and artificial origin (for instance, volcanic lava, mudflows, oil, polymer solutions and melts, suspensions, paints, honey, and grainy materials) called non-Newtonian fluids depends on the shear velocity of the flow. An adequate equation of state for these fluids is considered to be the Ostwald–de Waele power law, which relates the shear stress and the shear velocity [1–9]. This law contains two parameters: dynamic viscosity η_0 [$\mathrm{kg \cdot m^{-1} \cdot sec^{n-2}}$] and the index n, which takes the values $0 < n < 1$ for pseudoplastic fluids and $n > 1$ for dilatant fluids. In particular, mudflows ($0.1 \leqslant n \leqslant 0.4$) and oil ($n = 0.8$) are pseudoplastic fluids, whereas the lime–water mixture ($n = 1.47$) and saccharified honey ($n = 2.5$) are dilatant fluids. The value $n = 1$ corresponds to the Newtonian fluid.

Stability of thin layers of incompressible non-Newtonian fluids flowing down inclined (at an angle α) planes to small periodic perturbations was analyzed in [5–9]. It is shown that such perturbations are unstable for Ostwald numbers $\mathrm{O}_n > \mathrm{O}_n^*$ and stable for $\mathrm{O}_n < \mathrm{O}_n^*$, where $\mathrm{O}_n^* = (2n+1)^{n-1} n^{2-n} \cot\alpha$ is the critical Ostwald number corresponding to a fluid with an index n. Surface tension, which should be taken into account in considering very thin layers (films) does not change this critical value, but the range of wavenumbers of unstable perturbations becomes finite. The numerical results in [6–9] refer to the evolution of the profile of the free surface of layers of some non-Newtonian fluids for initial perturbations of two types, which are a localized triangular swelling of finite height and width and a smooth step that is not limited upstream. As the perturbations move, the profiles of the free surface become significantly different. For instance, the initial step that appeared on the surface of a layer of a pseudoplastic fluid is transformed into a series of jumps with a steep leading front with monotonically varying thickness of the fluid layer between these jumps. This structure is similar to steady roll waves studied by Ng and Mei [5], who found that steady solutions of the smoothed jump type do not exist in pseudoplastic fluids. In the present work, we study steady and unsteady waves of the type of a smoothed hydraulic jump, running on the surface of a layer of a dilatant fluid on an inclined plane. Some numerical solutions of such a problem can be found in [10].

Similar to [5–9], we write the initial equations in the following form:

$$
\begin{gathered}
\rho(u_t + u u_x + v u_y) = -p_x + \rho g \sin\alpha + (\sigma_{xx})_x + (\tau_{xy})_y, \\
\rho(v_t + u v_x + v v_y) = -p_y - \rho g \cos\alpha + (\tau_{yx})_x + (\sigma_{yy})_y, \\
u_x + v_y = 0.
\end{gathered}
\qquad (1)
$$

The coordinate x is directed along the inclined plane, the coordinate y is normal to it, u and v are the x- and y-components of velocity, σ and τ are the normal and tangential components of the stress tensor, α is the angle of the inclined plane, p is the pressure, and ρ is the density of the fluid. For non-Newtonian fluids, we have

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0021-8944/02/4302-0269 $27.00

$$\sigma_{xx} = 2\rho\nu_n A_n u_x, \qquad \sigma_{yy} = 2\rho\nu_n A_n v_y,$$

$$\tau_{xy} = \tau_{yx} = \rho\nu_n A_n(u_y + v_x), \qquad A_n = [2u_x^2 + 2v_y^2 + (u_y + v_x)^2]^{(n-1)/2},$$

where ν_n [$\mathrm{m}^2 \cdot \mathrm{sec}^{n-2}$] is the kinematic viscosity of the fluid with the index n.

We supplement system (1) by no-slip boundary conditions on the plane $y = 0$ and zero stresses, and also by the kinematic equation on the free surface $y = H(x,t)$. We introduce the following scales: L_0 and H_0 are the lengths along and normal to the inclined plane, $p_0 = \rho u_0^2$ is the pressure, $u_0 = (g\sin\alpha/\nu_n)^{1/n}(n/(2n+1))H_0^{(n+1)/n}$ is the longitudinal velocity, and $t_0 = L_0/u_0$ is the time. Assuming that the undisturbed thickness of the layer H_0 is much smaller than the characteristic length along the inclined plane L_0 ($\varepsilon \equiv H_0/L_0 \ll 1$), we determine the orders of the terms and integrate Eqs. (1) along the y coordinate from 0 to $H(x,t)$. As a result, we obtain equations averaged over the thickness of a thin layer of a non-Newtonian fluid on a rough inclined plane:

$$H_t + Q_x = 0; \tag{2}$$

$$Q_t + c_1(Q^2/H)_x = (c_2/\mathrm{O}_n)[(1 - H_x \cot\alpha)H - Q^n/H^{2n}]. \tag{3}$$

Here $H(x,t)$ is the function that describes the shape of the free surface, $Q(x,t) = \int\limits_0^H u\,dy$ is the flow rate of the fluid, $c_1 = (4n+2)/(3n+2)$, $c_2 = ((2n+1)/n)^n$, and $\mathrm{O}_n = H_0^n u_0^{2-n}/\nu_n$ is the Ostwald number.

We consider the possibility of existence of swellings on the free surface of the layer, which do not change their shape in a coordinate system moving with a constant velocity V. We assume that the functions H and Q depend on $\xi = x - Vt$ only and substitute the partial derivatives with respect to the x coordinate and time t by the derivative with respect to ξ: $\partial/\partial t = -V d/d\xi$, $\partial/\partial x = d/d\xi$. Integrating Eq. (2), we obtain the relation between the flow rate and the surface swelling $Q = VH + C$; substituting the latter into Eq. (3), we obtain a first-order ordinary differential equation that describes the shape of the free surface:

$$\Big[\mathrm{O}_n\Big((c_1 - 1)V^2 - \frac{c_1 C^2}{H^2}\Big) + c_2 H \cot\alpha\Big]\frac{dH}{d\xi} = c_2\Big(H - \frac{(VH + C)^n}{H^{2n}}\Big). \tag{4}$$

Here C is the constant of integration.

We seek the solution of Eq. (4) in the form of a smoothed hydraulic jump: $\lim\limits_{\xi\to-\infty} H(\xi) = H_1 = \mathrm{const}$ and $\lim\limits_{\xi\to\infty} H(\xi) = 1$. The latter equality means that the free-stream parameters $H_0 = Q_0 = 1$ as $\xi \to \infty$ are used as the scales of the layer thickness and fluid flow rate. In this case, we have $C = 1 - V$, and the equation for the wave profile becomes

$$\frac{dH(\xi)}{d\xi} = \frac{f(H)}{g(H)}, \tag{5}$$

where $f(H) = c_2[H - (V(H-1)+1)^n/H^{2n}]$ and $g(H) = \mathrm{O}_n[(c_1-1)V^2 - c_1(1-V)^2/H^2] + c_2 H \cot\alpha$.

For the above choice, the shape of the possible solution is $\lim\limits_{H\to H_1}(dH/d\xi) = 0$; therefore, we have $\lim\limits_{H\to H_1} f(H) = 0$. Hence, we can determine the wave velocity

$$V = (H_1^{(2n+1)/n} - 1)/(H_1 - 1). \tag{6}$$

It follows from formula (6) that, for high amplitudes H_1, the velocity is $V = H_1^{(n+1)/n}$; for strongly dilatant fluids ($n \gg 1$), the wave velocity is directly proportional to the amplitude H_1.

In a smoothed hydraulic jump, the profile is characterized by a negative derivative $dH/d\xi < 0$ everywhere except for the points $H = 1$ and $H = H_1$ where it vanishes. The sign of $dH/d\xi$ is determined by the signs of the functions $f(H)$ and $g(H)$. The calculations show that the function $f(H)$ is always negative for $1 < H < H_1$, and the sign of the function $g(H)$ depends on the jump amplitude. If H_1 is lower than a certain critical value $H_1^*(n)$, then the function $g(H)$ is positive, and the derivative $dH/d\xi$ is negative everywhere within the range of variation of the layer thickness from the amplitude value H_1 to the undisturbed value $H = 1$. This means that the swelling of the free surface monotonically decreases from H_1 to $H = 1$; hence, a continuous solution of the smoothed jump type exists. For $H_1 > H_1^*(n)$, the function $g(H)$ may become negative and the derivative $dH/d\xi$ may become positive. In this case, there is no continuous profile: the swelling of the free surface monotonically decreases from the amplitude

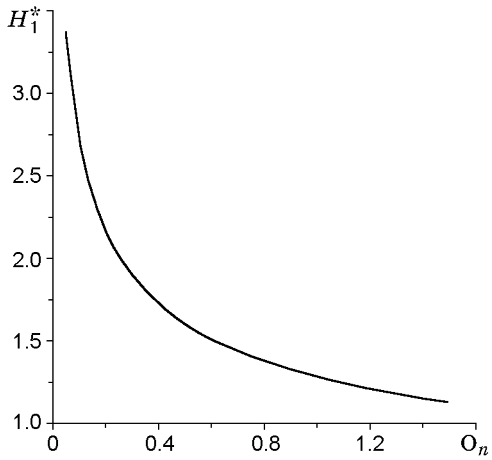

Fig. 1

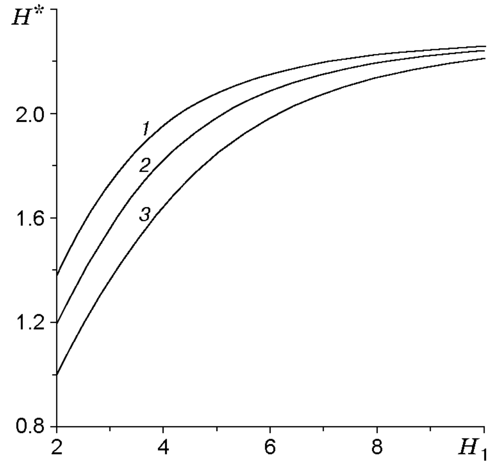

Fig. 2

Fig. 1. Critical amplitude of a steady wave versus the Ostwald number.

Fig. 2. Layer thickness behind the jump versus the steady-wave amplitude for $\mathrm{O}_n = 1$ (1), 0.5 (2), and 0.25 (3).

H_1 to the value $H = H^* > 1$, at which the derivative $dH/d\xi$ turns to infinity. The undisturbed level $H = 1$ can be reached only if there is a discontinuity in this place. However, such discontinuous solutions have a formal character, since at the points where $H \to H^*$, the main assumption of the model is violated, which reads that the thickness of the fluid layer is much smaller than the characteristic longitudinal scale of the problem. To determine the value of $H_1^*(n)$, we solve the equation $g(H) = 0$ for several increasing values of the amplitude H_1. If H_1 is a little higher than $H = 1$, then the real root of this equation is smaller than unity and, hence, is located outside the range of variation of the layer thickness for the solutions of the type considered. As H_1 increases, the real root of the equation $g(H) = 0$ tends monotonically to unity from the side of lower values. The point of transition of this root through unity corresponds to the critical amplitude H_1^*. For discontinuous solutions corresponding to the values $H_1 > H_1^*$, the layer thickness behind the jump is determined as the real root of the equation $g(H) = 0$.

All calculations were performed for a water solution of lime with an index $n = 1.47$. Figure 1 shows the critical amplitude of the jump H_1^* as a function of the Ostwald number O_n. It is seen that H_1^* decreases with increasing O_n. Figure 2 shows the layer thickness H^* behind the jump versus the amplitude H_1 for different values of O_n. The value of H^* increases monotonically for all values of O_n with increasing wave amplitude. For a fixed value of H_1, the layer thickness behind the jump increases with increasing O_n.

The continuous profiles of the free surface of the dilatant fluid layer in the region $1 \leqslant H_1 < H_1^*(n)$ can be found by solving numerically Eq. (5) with given parameters H_1, n, and O_n. Since the derivative $dH/d\xi$ vanishes at the point $H = H_1$, one has to move apart from this point and start calculations from the value of the sought function $H = H_1 - h$, where $h \ll H_1$ is a solution of Eq. (5) linearized near the singular point $H = H_1$. Such numerical solutions are smoothed jumps whose front width decreases with increasing amplitude. A similar analysis for pseudoplastic fluids ($n < 1$) shows that solutions of the smoothed hydraulic type do not exist. In this case, the solutions, which are stationary in a coordinate system moving with a certain constant velocity, have the form of periodic jags with steep leading fronts [5].

We consider unsteady waves propagating over the surface of a dilatant fluid with a swelling of the free surface $H_2(0, t) > 1$ and the corresponding flow rate and velocity of the fluid set at the left boundary of the computational domain. As in [5, 6], Eqs. (2) and (3) were solved numerically using an explicit finite-difference scheme. According to the calculations, there are two typical regimes, depending on whether the value of $H_2(0, t)$ at the boundary of the computational domain, which is used as the jump amplitude H_1, corresponds to continuous or discontinuous steady (in the moving coordinate system) solutions. For $H_2 < H_1^*$, a smoothed jump with an amplitude H_2 rapidly emerges; it propagates with a constant velocity over the surface of the undisturbed layer and does not change its shape. The value of velocity calculated by the displacement of the steepest point of the profile coincides with that obtained by Eq. (6) assuming that $H_1 = H_2(0, t)$. In particular, such a flow pattern is observed in calculations with

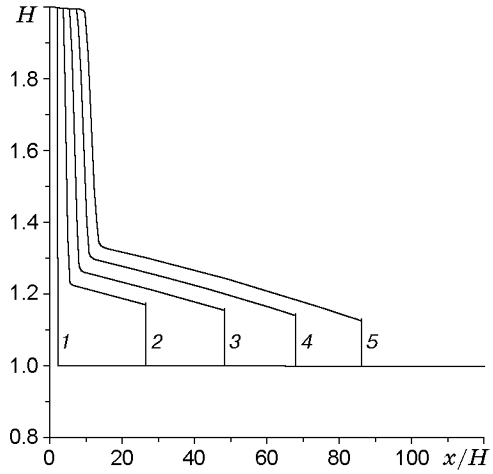

Fig. 3

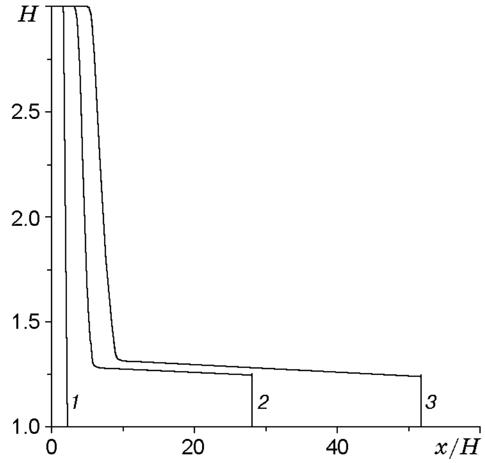

Fig. 4

Fig. 3. Profile of the free surface of the fluid layer ($O_n = 20$ and $H_2 = 2$) for $t = 0$ (1), 1.25 (2), 2.5 (3), 3.75 (4), and 5 (5).

Fig. 4. Profile of the free surface of the fluid layer ($O_n = 20$ and $H_2 = 3$) for $t = 0$ (1), 0.625 (2), and 1.25 (3).

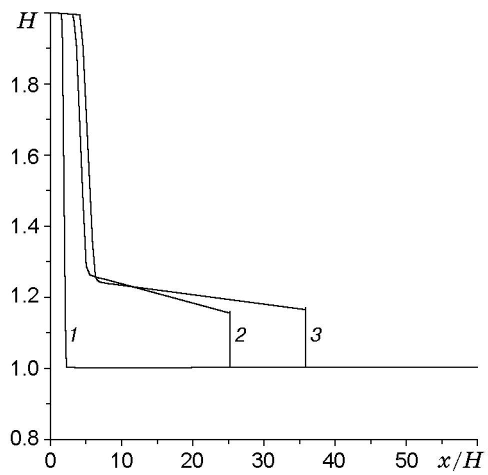

Fig. 5. Profiles of the free surface of the fluid layer ($t =$ 23.7 msec): curve 1 shows the initial profile; curves 2 and 3 refer to $O_n = 10$ and 20, respectively.

the parameters $O_n = 1$ and $H_2(0,t) = 1.2$, since the critical amplitude is $H_1^* = 1.28$ in this case.

If we increase O_n and choose $H_2(0,t)$ such that the corresponding formal steady (in the moving coordinate system) solution is discontinuous, the evolution of the flow is different: a rapidly moving precursor with a steep leading front is formed on the profile of the free surface (Fig. 3). Its velocity U decreases with time ($U = 20.4$, 18.2, 16.7, and 14.5 at the times $t = 1.25$, 2.5, 3.75, and 5.0, respectively). The velocity of the part of the profile behind the precursor is approximately constant and equal to $U_1 = 2.18$. At greater times, the precursor amplitude decreases asymptotically, and the profile becomes "diffusive". An increase in the amplitude of the free surface profile $H_2(0,t)$ with an unchanged value of O_n leads to an increase in the precursor velocity (Fig. 4). It follows from a comparison of curve 2 in Fig. 3 and curve 3 in Fig. 4, which show the profile shape at the same dimensionless time ($t = 1.25$).

The time scale used in passing to dimensionless variables is

$$t_0 = (H_0/(g\sin\alpha))^{1/2}((2n+1)/n)^{n/2}\mathrm{O}_n^{-1/2}.$$

If we choose a particular fluid with a certain value of the index n and fix the undisturbed thickness of the layer H_0 and the angle α of the inclined plane to the horizontal line, then the ratio of time scales corresponding to two Ostwald numbers is $t_0^{(1)}/t_0^{(2)} = (\mathrm{O}_n^{(2)}/\mathrm{O}_n^{(1)})^{1/2}$. It follows from here that an increase in O_n leads to a decrease in the time scale. The dimensionless time is determined as $\bar{t} = t/t_0$, where t is the dimensional (physical) time; hence, we have

$$\bar{t}^{(1)}/\bar{t}^{(2)} = (t^{(1)}/t^{(2)})(\mathrm{O}_n^{(1)}/\mathrm{O}_n^{(2)})^{1/2}.$$

It follows from this formula that one dimensionless time $\bar{t}^{(1)} = \bar{t}^{(2)}$ in calculations with two values of O_n corresponds to different instants of physical time $t^{(1)}/t^{(2)} = (\mathrm{O}_n^{(2)}/\mathrm{O}_n^{(1)})^{1/2}$, whereas one moment of physical time $t^{(1)} = t^{(2)}$ corresponds to different instants of dimensionless time $\bar{t}^{(1)}/\bar{t}^{(2)} = (\mathrm{O}_n^{(1)}/\mathrm{O}_n^{(2)})^{1/2}$. If $\mathrm{O}_n^{(2)} > \mathrm{O}_n^{(1)}$, then $t^{(2)} < t^{(1)}$ and $\bar{t}^{(2)} > \bar{t}^{(1)}$. Figure 5 shows the profiles of the free surface of the water solution of lime for an identical amplitude and two values of O_n at the same instant of physical time $t = 23.7$ msec. The flow develops faster at higher Ostwald numbers.

Thus, roll waves of the smoothed hydraulic jump type with an unchanged shape can propagate over the surface of thin layers of dilatant fluids moving along inclined planes, if their amplitude is lower than a certain critical value depending on the index of the fluid and Ostwald number. For higher amplitudes, such steady waves do not exist, and unsteady waves have a more complicated structure.

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